

Fractional derivatives in Dengue epidemics*

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Abstract

We introduce the use of fractional calculus, i.e., the use of integrals and derivatives of non-integer (arbitrary) order, in epidemiology. The proposed approach is illustrated with an outbreak of dengue disease, which is motivated by the first dengue epidemic ever recorded in the Cape Verde islands off the coast of west Africa, in 2009. Numerical simulations show that in some cases the fractional models fit better the reality when compared with the standard differential models. The classical results are obtained as particular cases by considering the order of the derivatives to take an integer value.

Keywords: fractional calculus, Riemann–Liouville derivatives, epidemiological model, dengue disease, numerical approximation.

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1 Introduction

The incidence of dengue has grown intensely in the last decades and, according to the World Health Organization, about 40% of world's population is now at risk. This global pandemic is attributed to the unprecedented population growth, the rising level of urbanization without adequate domestic water supplies, increasing movement of the virus between humans (due to tourism, migration, or international trade), and lack of effective mosquito control [1]. Dengue virus is transmitted to humans through the bite of infected *Aedes* mosquitoes, specially *Aedes Aegypti*. Once infected, a mosquito remains infected for life, transmitting the virus to susceptible individuals during feed. Without a vaccine, vector control remains the only available strategy against dengue.

Appropriate mathematical models can give a deeper insight into the mechanism of disease transmission [2, 3, 4]. Typically, epidemiologic models are formulated with classical derivatives of integer order. Here we propose the use of generalized fractional derivatives. Fractional calculus (calculus of non-integer order), in spite of its long history as a pure branch of mathematics, only

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recently has shown to be useful as a practical tool [5]. In this paper we claim that fractional calculus provides an interesting modeling technique in the context of epidemiology.

We begin by considering a simple epidemiological model that represents an episode of dengue disease. The rest of the paper is then dedicated to introduce the notion of fractional derivative in the sense of Riemann–Liouville, reformulating the dynamics of the classical model in terms of fractional derivatives, and finally applying a recent approximate technique to obtain numerical solutions to the fractional model. Numerical simulations show that models with fractional derivatives may provide a better description of reality when compared with the standard ones.

2 Epidemiological model

In this paper we assume that the host population is divided into three classes: susceptible, $S_h(t)$, individuals who can contract the disease; infected, $I_h(t)$, individuals capable of transmitting the disease to others; and resistant, $R_h(t)$, individuals who have acquired immunity at time t . The total number of hosts is constant, i.e., $N_h = S_h(t) + I_h(t) + R_h(t)$. Similarly, we also have two compartments for the mosquito: $S_m(t)$ and $I_m(t)$ with $N_m = S_m(t) + I_m(t)$. The model is described by the system of differential equations

$$\begin{cases} \frac{dS_h}{dt}(t) = \mu_h N_h - (B\beta_{mh} \frac{I_m}{N_h} + \mu_h) S_h \\ \frac{dI_h}{dt}(t) = B\beta_{mh} \frac{I_m}{N_h} S_h - (\eta_h + \mu_h) I_h \\ \frac{dR_h}{dt}(t) = \eta_h I_h - \mu_h R_h \\ \frac{dS_m}{dt}(t) = \mu_m N_m - (B\beta_{hm} \frac{I_h}{N_h} + \mu_m) S_m \\ \frac{dI_m}{dt}(t) = B\beta_{hm} \frac{I_h}{N_h} S_m - (\mu_m) I_m \end{cases} \quad (1)$$

subject to given initial conditions $S_h(0)$, $I_h(0)$, $R_h(0)$, $S_m(0)$ and $I_m(0)$. The recruitment rate of human and vector populations are denoted as $\mu_h N_h$ and $\mu_m N_m$, respectively. The natural death rate for humans and mosquitoes is described by the parameters μ_h and μ_m , respectively. We assume that B is the average daily biting (per day) of the mosquito whereas β_{mh} and β_{hm} are related to the transmission probability (per bite) from infected mosquitoes to humans and vice versa. The recovery rate of the human population is denoted by η_h .

3 Fractional Calculus

Fractional calculus was originated at the end of the seventeenth century and consists in the study of derivatives and integrals of arbitrarily real or even complex order. Recently, fractional calculus is experiencing an intensive progress in both theory and applications [5, 6]. Main claim is that a fractional model can give a more realistic interpretation of natural phenomena. Among several different definitions that can be found in the literature for fractional derivative, one of the most popular is the Riemann–Liouville derivative.

Definition 1 Let $x(\cdot)$ be an absolutely continuous function in $[a, b]$ and $0 \leq \alpha < 1$. The (left) Riemann–Liouville fractional derivative of order α , ${}_a D_t^\alpha$, is given by

$${}_a D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} x(\tau) d\tau, \quad t \in [a, b].$$

We reformulate system (1) using fractional derivatives. To this end, we simply substitute the first-order derivatives by Riemann–Liouville derivatives of order α :

$$\begin{cases} {}_0 D_t^\alpha S_h(t) = \mu_h N_h - (B\beta_{mh} \frac{I_m}{N_h} + \mu_h) S_h \\ {}_0 D_t^\alpha I_h(t) = B\beta_{mh} \frac{I_m}{N_h} S_h - (\eta_h + \mu_h) I_h \\ {}_0 D_t^\alpha R_h(t) = \eta_h I_h - \mu_h R_h \\ {}_0 D_t^\alpha S_m(t) = \mu_m N_m - (B\beta_{hm} \frac{I_h}{N_h} + \mu_m) S_m \\ {}_0 D_t^\alpha I_m(t) = B\beta_{hm} \frac{I_h}{N_h} S_m - (\mu_m) I_m. \end{cases} \quad (2)$$

Note that when $\alpha \rightarrow 1$ the fractional system (2) reduces to (1). Given a certain reality, our goal is to find the order α that makes the model more realistic. As we shall see in the next section, the best value of α in system (2) is in general different from one, i.e., the classical model (1) is often not the best choice.

4 Numerical simulations

Similarly to the classical theory of differential equations, there are no general methods to solve systems of fractional differential equations analytically. Furthermore, the fractional case is believed to be more difficult to handle even numerically [7]. Recently, an approximation based on a continuous expansion formula for the Riemann–Liouville fractional derivative has been proposed in [8] and further improved in [9]. The basic idea is to express fractional terms by means of a series involving integer-order derivatives. The immediate result is that one can then use classical methods to obtain an approximate solution to the original fractional problem. With the same assumptions of Definition 1, the approximation for ${}_a D_t^\alpha x(\cdot)$ is

$${}_a D_t^\alpha x(t) \simeq A(\alpha, N)(t-a)^{-\alpha}x(t) + A'(\alpha, N)(t-a)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^N C(\alpha, p)(t-a)^{1-p-\alpha}V_p(t), \quad (3)$$

where

$$\begin{cases} \dot{V}_p(t) = (1-p)(t-a)^{p-2}x(t) \\ V_p(a) = 0, \quad p = 2, 3, \dots, N, \end{cases}$$

and $A = A(\alpha, N)$, $A' = A'(\alpha, N)$ and $C_p = C(\alpha, p)$ are given by

$$\begin{aligned} A(\alpha, N) &= \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^N \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right], \\ A'(\alpha, N) &= \frac{1}{\Gamma(2-\alpha)} \left[1 + \sum_{p=1}^N \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right], \\ C(\alpha, p) &= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \frac{\Gamma(p-1+\alpha)}{(p-1)!}. \end{aligned}$$

Using (3), we approximate system (2) by a system of ordinary differential equations in which, subject to the order of approximation N , new variables of the form $V_P(\cdot)$ will appear:

$$\begin{cases} \frac{d\tilde{S}_h}{dt}(t) = \left[\mu_h N_h - (B\beta_{mh} \frac{\tilde{I}_m}{N_h} + \mu_h) \tilde{S}_h - At^{-\alpha} \tilde{S}_h + \sum_{p=2}^N C_p t^{1-p-\alpha} V_p^{S_h}(t) \right] A'^{-1} t^{\alpha-1} \\ \frac{dV_p^{S_h}}{dt}(t) = (1-p)(t-a)^{p-2} \tilde{S}_h(t), \quad p = 2, 3, \dots, N \\ \frac{d\tilde{I}_h}{dt}(t) = \left[B\beta_{mh} \frac{\tilde{I}_m}{N_h} \tilde{S}_h - (\eta_h + \mu_h) \tilde{I}_h - At^{-\alpha} \tilde{I}_h + \sum_{p=2}^N C_p t^{1-p-\alpha} V_p^{I_h}(t) \right] A'^{-1} t^{\alpha-1} \\ \frac{dV_p^{I_h}}{dt}(t) = (1-p)(t-a)^{p-2} \tilde{I}_h(t), \quad p = 2, 3, \dots, N \\ \frac{d\tilde{R}_h}{dt}(t) = \left[\eta_h \tilde{I}_h - \mu_h \tilde{R}_h - At^{-\alpha} \tilde{R}_h + \sum_{p=2}^N C_p t^{1-p-\alpha} V_p^{R_h}(t) \right] A'^{-1} t^{\alpha-1} \\ \frac{dV_p^{R_h}}{dt}(t) = (1-p)(t-a)^{p-2} \tilde{R}_h(t), \quad p = 2, 3, \dots, N \\ \frac{d\tilde{S}_m}{dt}(t) = \left[\mu_m N_m - (B\beta_{hm} \frac{\tilde{I}_h}{N_h} + \mu_m) \tilde{S}_m - At^{-\alpha} \tilde{S}_m + \sum_{p=2}^N C_p t^{1-p-\alpha} V_p^{S_m}(t) \right] A'^{-1} t^{\alpha-1} \\ \frac{dV_p^{S_m}}{dt}(t) = (1-p)(t-a)^{p-2} \tilde{S}_m(t), \quad p = 2, 3, \dots, N \\ \frac{d\tilde{I}_m}{dt}(t) = \left[B\beta_{hm} \frac{\tilde{I}_h}{N_h} \tilde{S}_m - (\mu_m) \tilde{I}_m - At^{-\alpha} \tilde{I}_m + \sum_{p=2}^N C_p t^{1-p-\alpha} V_p^{I_m}(t) \right] A'^{-1} t^{\alpha-1} \\ \frac{dV_p^{I_m}}{dt}(t) = (1-p)(t-a)^{p-2} \tilde{I}_m(t), \quad p = 2, 3, \dots, N. \end{cases} \quad (4)$$

System (4) is solved numerically using the following values that are inspired in the outbreak occurred in 2009 in Cape Verde: $N_h = 56000$, $B = 0.7$, $\beta_{mh} = 0.36$, $\beta_{hm} = 0.36$, $\mu_h = 1/(71 \times 365)$,

$\eta_h = 1/3$, $\mu_m = 1/10$, $m = 3$, and $N_m = m \times N_h$. The initial conditions for the system (1) of ordinary differential equations are: $S_h(0) = Nh - 216$, $I_h(0) = 216$, $R_h(0) = 0$, $S_m(0) = N_m$ and $I_m(0) = 0$. These values, together with a zero initial condition for $V_p^{S_h}$, $V_p^{I_h}$, $V_p^{R_h}$, $V_p^{S_m}$, and $V_p^{I_m}$, make system (4) to be an ordinary initial value problem that can be treated numerically. Figure 1 shows the solution to the systems (1) and (4) with respect to the variable I_h . The system (4) is solved with $N = 7$ and different values of α . In order to compare both fractional ($\alpha \in [0, 1)$) and

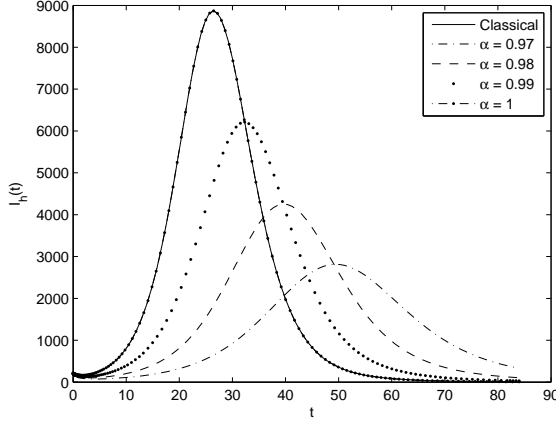


Figure 1: solution to the classical model (1) versus the solution to the fractional model (2) with different values of α .

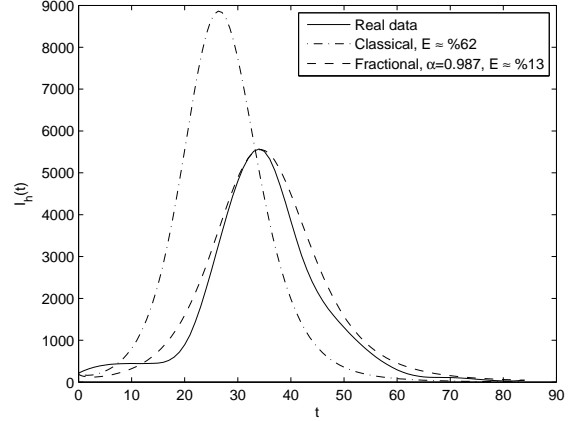


Figure 2: real data versus the solution to the classical model (1) and the solution to fractional model (2) with $\alpha = 0.987$.

classical ($\alpha = 1$) models, we include the statistical information from Cape Verde in Figure 2. The best model is, in our case, the fractional model with $\alpha = 0.987$, which corresponds a percentage error of thirteen. In contrast, the percentage error associated with the classical model is sixty two.

5 Conclusions

Describing the reality through a mathematical model, usually a system of differential equations, is a hard task that has an inherent compromise between simplicity and accuracy. In this paper we consider a very basic model to dengue epidemics. It turns out that, in general, this basic/classical model does not provide enough good results. In order to have better results, that fit the reality, more specific and complicated set of differential equations have been investigated in the literature — see [10, 11, 12] and references therein. Here we propose a completely new approach to the subject. We keep the simple model and substitute the usual (local) derivatives by (nonlocal) fractional differentiation. The use of fractional derivatives allow us to model memory effects, and result in a more powerful approach to epidemiological models: one can then design the order α of fractional differentiation that best corresponds to reality. The classical case is recovered by taking the limit when α goes to one. Our investigations show that even a simple fractional model may give surprisingly good results. However, the transformation of a classical model into a fractional one makes it very sensitive to the order of differentiation α : a small change in α may result in a big change in the final result.

The present work can be extended in several ways: by fractionalizing more sophisticated models; by considering different orders of fractional derivatives for each one of the state variables, i.e., models of non-commensurate order.

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